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Simple Groups as far as Order 660.

By F. N. Cole, Ph. D.

1. In an earlier paper[*] I have shown that the orders of simple groups between the limits 200 and 500 are restricted to two possibilities, 360 and 432. In the following it is shown that the latter of these two orders is inadmissible, and that the former furnishes only one type of a simple group. On continuing the census as far as order 660, two other simple groups present themselves, of orders 504 and 660 respectively. Of these the second is familiar from the theory of the elliptic modular functions. That of order 504, however, seems hardly to have been recognized hitherto.

The list of simple groups of compound orders to order 660, as determined by Dr. Hölder and myself, consists of one type for each of the orders 60, 168, 360, 504, and 660.

I.—*First Reduction of Possible Orders.*

2. A preliminary survey of the numbers from 500 to 660 shows that the orders of simple groups between these limits are restricted to the following 13 possibilities:

$504 = 2^3.3^2.7,$

$520 = 2^3.5\,.13,$

$525 = 3\,.5^2.7,$

$528 = 2^4.3\,.11\,,$

$540 = 2^2.3^3.5\,,$

$546 = 2\,.3\,.7.13,$

$552 = 2^3.3\,.23\,,$

$560 = 2^4.5\,.7,$

$576 = 2^6.3^2,$

$612 = 2^2.3^2.17,$

$616 = 2^3.7\,.11,$

$630 = 2\,.3^2.5\,.7,$

$660 = 2^2.3\,.5\,.11.$

To these we add the orders left unconsidered in the former paper,

$$360 = 2^3.3^2.5, \quad 432 = 2^4.3^3.$$

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[*] Amer. Jour., Vol. XIV, pp. 378-88.

3. The following four cases are immediately rejected:

520, 546, 552, 616,

inasmuch as the subgroups indicated by Sylow's theorem in each case would furnish a number of operations exceeding the order of the group. Thus a simple group of order 520 must contain 40 subgroups of order 13 and 26 of order 5; one of order 546 must contain 78 subgroups of order 7 and 14 of order 13; one of order 552 must contain 24 subgroups of order 23 and 46 of order 3; finally, one of order 616 must contain 56 subgroups of order 11 and 8 of order 7, leaving in this case only 8 operations remaining, from which only *one* subgroup of order 8 could be formed.

4. Of the 11 remaining cases the following 5 also present no difficulty:

525, 528, 540, 560, 576.

A simple group of order 525 must contain 15 subgroups of order 7. The isomorphic substitution group of 15 letters would contain 15 subgroups of order 35 affecting 14 letters each. But a group of order 35 is cyclical, and its generating substitutions must here consist of one cycle of 5 letters with one cycle of 7 letters. The group would therefore contain circular substitutions of 5 and of 7 letters, which are here impossible.

A simple group of order 528 must contain 12 subgroups of order 11. The isomorphic substitution group of 12 letters would contain 12 subgroups of order 44 affecting 11 letters each. The latter contain each a self-conjugate subgroup of order 11, and being therefore transitive, each contains subgroups of order 4 affecting 10 letters. The substitutions of the latter must be even, and none of them affect less than 5 letters. Then it is readily seen that they all affect 8 letters. But no such substitution can transform the circular substitutions of order 11 into any of their powers, as must here be the case.

A simple group of order 540 must contain 36 subgroups of order 5. The isomorphic substitution group of 36 letters would contain 36 subgroups of order 15 affecting 35 letters each. A group of order 15 is cyclical. Its substitutions of order 5 must here affect all the 35 letters, otherwise there would not be 36 *distinct* subgroups of order 5. Consequently the substitutions of order 15 must here consist of cycles of 15 letters with cycles of 5 letters, and we may have either 1 cycle of 15 letters with 4 of 5 letters or 2 cycles of 15 letters with 1 of 5 letters. But the group of order 15 contains only one subgroup of order 3, which will here affect 15 letters or 30 letters. The number of its conjugates would therefore be $\frac{36}{21}$ or $\frac{36}{6}$, both of which are impossible.

A simple group of order 560 must contain 8 subgroups of order 7. But a substitution group of order 560 and degree 8 would contain circular substitutions of order 5, and would accordingly be non-primitive or four-fold transitive, both of which possibilities are here excluded.

A simple group of order 576 would contain 9 subgroups of order 64 which, in the isomorphic substitution group of 9 letters, would affect 8 letters each. The substitutions of these subgroups must all be of order 2 or 4, for a circular substitution of 8 letters is odd. Rejecting all odd substitutions and those which affect less than 5 letters, we have left only such as affect each 8 letters. The 9 subgroups of order 64 then furnish $9.63 = 567$ distinct substitutions, leaving only 9, which can furnish only *one* subgroup of order 9.

5. Fairly simple considerations also suffice to dispose of the orders

432, 612, 630.

A simple group of order 432 must contain 16 subgroups of order 27 which, in the isomorphic substitution group of 16 letters, would affect 15 letters each. These subgroups of order 27 again contain subgroups of order 9, which are always commutative with the 27 operations of the group in which they occur, and which are therefore commutative within the entire group with a multiple of 27 operations. The number of conjugates within the entire group of such a subgroup of order 9 is therefore a divisor of $\frac{432}{27} = 16$. The only possibility here is 16 itself, since 8! is not divisible by 432. Consequently every subgroup of order 9 is commutative with exactly 27 operations, and therefore occurs in only one group of order 27. Then any two groups of order 27 can have only 1 or 3 operations in common, or, in other words, all but 1 or all but 3 of the substitutions of a group of order 27 affect all the 15 letters. In the former case the 16 subgroups of order 27 furnish $16.26 = 416$ distinct operations, leaving only 16, which can give only *one* subgroup of order 16. In the latter case the 3 substitutions of any group of order 27 which affect less than 15 letters form a cyclical group affecting 3, 6, 9 or 12 letters, and having therefore $\frac{16}{13}$, $\frac{16}{10}$, $\frac{16}{7}$, or $\frac{16}{4}$ conjugates in the entire group, all of which suppositions are here impossible.

A simple group of order 612 must contain 18 subgroups of order 17. The isomorphic substitution group of 18 letters would contain 18 subgroups of order 34 affecting 17 letters each. The latter groups must here be non-cyclical; each of them contains 1 subgroup of order 17 and 17 subgroups of order 2, the latter affecting 16 letters each. There are no other substitutions of order 2 or of order 4 in the group. For if such were present they must affect 18 letters each. If of order 2 they would then be odd, and if of order 4, their squares must affect 16 letters, and they must consist therefore of 4 cycles of 4 letters with 1 cycle of two letters, being again odd. The operations of order 2 contained in the entire group are therefore all conjugate, and their number is $\frac{18}{2} \cdot 17 = 153$. The corresponding group of 612 substitutions of 153 letters would contain 153 subgroups of order 4 affecting 152 letters each. The actual substitutions of the latter groups are taken entirely from the 153 conjugate substitutions of order 2. These must all leave the same number of letters unchanged. If this number is x, the total number of substitutions of order 2 would be $\frac{153}{x} \cdot 3$. Hence $x = 3$, and every substitution of order 2 would affect 150 letters, and would therefore be odd.

A simple group of order 630 must contain either 21 or 126 subgroups of order 5. In the latter case the isomorphic substitution group of 126 letters would contain 126 subgroups of order 5 affecting 125 letters each, while all the other substitutions of the group would affect all the 126 letters. Among these there must be substitutions of order 2, which would then be odd. Accordingly a simple group of order 630 must contain exactly 21 subgroups of order 5. The isomorphic substitution group of 21 letters would then contain 21 subgroups of order 30 affecting 20 letters each. A group of order 30 contains a (cyclical) subgroup of order 15, the generating substitutions of which must here consist of 1 cycle of 15 letters and 1 of 5 letters. But a group of order 30 contains only one subgroup of order 3, which in the present case would affect 15 letters and would therefore have in the entire group $\frac{21}{6}$ conjugates.

II.—*The Simple Group of Order* 360.

6. A simple group of order 360 must contain either 6 or 36 subgroups of order 5. The former number is at once excluded, since the alternating substitution group of 6 letters contains 36 subgroups of order 5. Also the number of subgroups of order 9 is either 10 or 40. It will appear that only the former case actually occurs.

If 40 subgroups of order 9 were present, these could not be cyclical, for then they would furnish 40.6 distinct operations of order 9 in addition to the 36.4 operations of order 5 which are certainly present.

The 40 subgroups of order 9 must therefore contain each 4 subgroups of order 3. These 160 subgroups could not all be different, for then they would furnish 320 distinct operations. On being transformed by any one of the operations of order 3, the 40 subgroups of order 9 would be permuted among themselves, and $3x + 1$ of them would be left unchanged. All those which contain the transforming operation of order 3 will certainly be left unchanged. And this is not the case with any of the others, for then the group of order 360 would contain subgroups of order 27. Accordingly some subgroup of order 3 is contained in 4, 7, 10, ... of the subgroups of order 9, and as it is commutative with all the operations of the latter, it is commutative with a subgroup whose order is a multiple of 4.9, 7.9, 10.9, ... Then the number of its conjugates in the entire group is a divisor of $\frac{360}{4.9}$, $\frac{360}{7.9}$, ... The only admissible case is the first one. A subgroup of order 3 which occurs in more than one subgroup of order 9 occurs in exactly 4 of the latter, and has exactly 10 conjugates. It follows at once that

A simple group of order 360 *can always be expressed as a transitive substitution group of* 10 *letters.*

This group of 10 letters contains 10 conjugate subgroups of order 36 affecting 9 letters each. These contain each 1 or 4 subgroups of order 9. If these are transitive, the groups of order 36 contain each 9 subgroups of order 4 affecting 8 letters each. The substitutions of the latter being even, it is readily seen that every one of them, except identity, affects 8 letters. The group of order 36 would then contain 27 substitutions of order 2 and 4 and consequently only 1 subgroup of order 9.

On the other hand, of intransitive groups of order 9 affecting 9 letters or less and not containing any circular substitution of 3 letters, there is only one type, its 4 subgroups of order 3 being, for example,

$$\{(abc)(def)\},\ \{(abc)(ghi)\},\ \{(def)(gih)\},\ \{(abc)(dfe)(gih)\}.$$

Now, if the group of order 360 contained 40 subgroups of order 9, each subgroup of order 36 would contain 4 of these, and they must have, as shown above, one common subgroup of order 3. But it is immediately found on trial that it is impossible to construct more than 2 such groups of the preceding type. Also, if there were only 10 subgroups of order 9 in all, and if these were of the preceding type, the total number of subgroups of order 3 affecting 6 letters would be $\frac{10}{4}.3$. We have then the following result:

A simple group of order 360 *contains only* 10 *subgroups of order* 9, *and can be expressed as a doubly transitive substitution group of* 10 *letters.*

The transitive subgroups of order 36 which affect 9 letters contain, as just shown, 1 subgroup of order 9, all of whose substitutions except identity affect all the 9 letters, and 9 subgroups of order 4, all of whose substitutions except identity affect 8 letters. The subgroup of order 9 cannot be cyclical, for the 3 substitutions affecting a particular set of 8 letters cannot all transform a cycle of 9 letters into its powers, as is readily seen. Consequently the subgroup of order 9 contains 4 subgroups of order 3; for example,

$$\{(abc)(def)(ghi)\},\ \{(adg)(beh)(cfi)\},\ \{(aei)(bfg)(cdh)\},\ \{(afh)(bdi)(ceg)\}.$$

It will be clear, from an inspection of this group, that no substitution with 4 cycles of 2 letters each can transform any one of the subgroups of order 3 into another, and the only substitution of order 2 which leaves i, for example, unchanged and transforms each subgroup of order 3 into itself is

$$(ae)(bd)(cf)(gh).$$

Accordingly, the subgroups of order 4 which leave 2 letters unchanged are *cyclical*, and with the same notation as above, the group which leaves i and j unchanged may be assumed to be

$$\{(abed)(chfg)\}.$$

A simple group of order 360, as expressed in 10 letters, contains then 135 substitutions affecting 8 letters and 80 affecting 9 letters. These with the 144 substitutions of order 5 and identity make up the entire group. It only remains to be shown that the group can be constructed in essentially only one way.

There are 36 subgroups of order 5, and each of these is therefore transformed into itself by 10 substitutions. Of these, 5 must be selected from the substitutions of order 2. Moreover, the latter being all conjugate, every one of them transforms some subgroup of order 5 into itself, and therefore transforms some substitution of order 5 into its 4th power. For the particular substitution of order 2,

$$(ae)(bd)(cf)(gh),$$

the corresponding substitution of order 5 must contain i and j in separate cycles. Moreover, if any of the letters a, b, e, d occur in the same cycle with i, a transformation by a power of $(abed)(chfg)$ will give a substitution of order 5 containing a and i in the same cycle and still transformed into its 4th power by $(ae)(bd)(cf)(gh)$. In a proper power of the substitution of order 5, a will follow i. We have, therefore, as typical forms for the cycle containing i only,

$$(iabde),\quad (iahge),\quad (iacfe),\quad (icghf),$$
$$(iadbe),\quad (iaghe),\quad (iafce),\quad (ichgf).$$

The first two and the last two cases are to be rejected. For these the cycle containing i is transformed into one of its powers by the substitution $(abed)(chfg)$. If the second cycle in each case is transformed at the same time into the same power, then the group of order 5 which the substitution of order 5 generates is transformed into itself by 20 substitutions, which is not permissible. And otherwise a proper combination of the substitution of order 5 with its transformed substitution would affect only 5 letters, which is also impossible.

Again, if we multiply the 4 remaining types at the left by $(aie)(bgf)(chd)$, we obtain substitutions which do not affect a or i, and which therefore consist of 4 cycles of 2 letters or 2 cycles of 4 letters. For portions of these substitutions we have

$$(beh\,..)(cg\,....),\quad (gec\,..)(df\,....),$$
$$(ceg\,..)(bh\,....),\quad (def\,..)(gc\,....),$$

and considering the various forms of the second cycle of 5 letters, we find only the following possibilities for the typical substitutions of order 5:

$$(iahge)(jbcfd),\quad (iacfe)(jgdbh),$$
$$(iaghe)(jcbdf),\quad (iafce)(jdghb).$$

Of these, the first and the fourth cases lead to rejected types; for we have

$$(acb)(dfe)(gih).(iahge)(jbcfd) = (afig)(bhej),$$
$$[(afig)(bhej)]^2.(ae)(bd)(cf)(gh) = (iedba)(jgefh),$$

and

$$(agd)(bhe)(cif).(iafce)(jdghb) = (ahic)(dfej),$$
$$[(ahic)(dfej)]^2.(ae)(bd)(cf)(gh) = (iebda)(jcghf).$$

The two remaining cases are identical. For

$$(abc)(def)(ghi).(iaghe)(jcbdf) = ihad.jcge,$$
$$[(ihad)(jcge)]^2.(ae)(bd)(cf)(gh) = (iefca)(jhbdg).$$

There is then only one simple group of order 360. *This is therefore isomorphic with the alternating substitution group of* 6 *letters.*

III.—*The Simple Group of Order* 504.

A simple group of order 504 must contain either 8 or 36 subgroups of order 7. In the former case the isomorphic substitution group of 8 letters would contain subgroups of order 63 affecting 7 letters. The latter would contain subgroups of order 9, which would here include circular substitutions of 3 letters. There must, therefore, be 36 subgroups of order 7.

The isomorphic substitution group of 36 letters contains 36 subgroups of order 14 affecting 35 letters each. These contain subgroups of order 7 which must affect all the 35 letters, since otherwise there would not be 36 distinct subgroups of order 7. If the groups of order 14 were cyclical, their generating substitutions, being even, must consist of 2 cycles of 14 letters and 1 cycle of 7 letters. Such a group would contain one substitution of order 2 affecting 28 letters, and having therefore in the entire group $\frac{36}{8}$ conjugates. Accordingly, the groups of order 14 are non-cyclical, and each of them contains, beside the subgroup of order 7, 7 conjugate substitutions of order 2, which transform the substitutions of order 7 into their 6th powers. The substitutions of order 2 might leave one letter of each of the 5 cycles of a substitution of order 7 unchanged and permute the 4 remaining letters of each cycle in pairs, or they might interchange the letters of one or of two pairs of cycles. It appears that an *even* substitution of order 2 must leave exactly 4 letters unchanged. The total number of conjugate substitutions of this type is therefore $\frac{36}{4}.7 = 63$.

Of subgroups of order 9 there must be either 7 or 28, and it appears at once that only the latter number is admissible. The isomorphic substitution group of 28 letters contains 28 subgroups of order 18 affecting 27 letters. Each of these subgroups of order 18 contains a single subgroup of order 9. If all the substitutions of the latter, except identity, affect all the 27 letters, these groups would furnish $28.8 = 224$ substitutions of order 3 or 9, and these with the $36.6 = 216$ of order 7, the 63 of order 2, and identity would make up the entire group.

If any of the substitutions of a group of order 9 affect less than 27 letters, the group cannot be cyclical. For then the subgroup affecting less than 27 letters could only be of order 3 and affect 9 or 18 letters, and such a subgroup would have $\frac{28}{19}$ or $\frac{28}{10}$ conjugates in the entire group. Again, if the groups of order 9 were non-cyclical, a subgroup which affected less than 27 letters could affect only 24, 21, 18, ... letters. It would then occur in 4, 7, 10, ... of the groups of order 9, and would therefore be commutative with a multiple of 4.9, 7.9, 10.9, ... substitutions of the entire group. Then its number of conjugates in the latter would be a divisor of $\frac{504}{4.9}, \frac{504}{7.9}, \ldots.$ The only admissible possibility is that it should occur in 4 groups of order 9 and have 14 conjugates.

The isomorphic substitution group of 14 letters would contain 14 subgroups of order 36, each including 4 non-cyclical subgroups of order 3. The latter have a subgroup of order 3 in common, and furnish beside this $4.3 = 12$ other subgroups of order 3. Among the remaining substitutions of the groups of order 36 must be contained the 63 conjugate substitutions of order 2. These, being even, must affect either 8 or 12 letters. If there are h of them in each group, the number of their conjugates is $63 = \frac{14h}{6}, \frac{14h}{2}$; hence $h = 27, 9$. Only the latter case is possible, and the groups of order 36 are then complete.

From the preceding considerations we deduce the result that

In a simple group of order 504 *the operations of order* 2 *are all conjugate, and their number is* 63.

The isomorphic substitution group of 63 letters contains 63 subgroups of order 8 affecting 62 letters each. These groups contain the 63 substitutions of order 2, the existence of which has just been demonstrated. If h of these occur in each group of order 8, and if each of them leaves k letters unchanged, their total number is $63 = \frac{63h}{k}$. Consequently $h = k$. But as $h < 8$ and the substitutions of order 2 are even, we can only take $k = 3, 7$.

In the latter case the groups of order 8 are made up entirely from the substitutions of order 2. These are all commutative with each other, and as each of them occurs in 7 of the groups of order 8, each is commutative with more than 8 substitutions, and has therefore less than 63 conjugates, *unless the* 7 *groups in which it occurs all coincide, so that there are only* 9 *distinct groups of order* 8.

Again, if $k = 3$, each group of order 8 contains 4 as yet unidentified substitutions. These could only be of order 4, and as their squares, being of order 2, must affect 60 letters, they must consist of 15 cycles of 4 letters and 1 cycle of 2 letters. But then they affect 62 letters, and the entire group would contain $63.4 = 252$ of them, which is impossible.

Accordingly a simple group of order 504 *contains exactly* 9 *subgroups of order* 8.

Expressed as a substitution group of 9 letters, the group contains circular substitutions of 7 letters, and is therefore triply transitive. The 9 doubly transitive subgroups which affect 8 letters each are of order 56, and each of them contains 8 circular subgroups of order 7 and a single subgroup of order 8. The latter, as already shown, contains, beside identity, 7 substitutions of order 2 which here affect 8 letters each. The substitutions of order 7 transform the 7 substitutions of order 2 in a cycle.

For the group of order 8 we may take

$$\begin{aligned} 1,\ & (ab)(cd)(ef)(gh), \\ & (ac)(bd)(eg)(fh), \\ & (ad)(bc)(eh)(fg), \\ & (ae)(bf)(cg)(dh), \\ & (af)(be)(ch)(dg), \\ & (ag)(bh)(ce)(df), \\ & (ah)(bg)(cf)(de), \end{aligned}$$

and for one of the substitutions of order 7,

$$\sigma = (bcedghf).$$

The substitution σ transforms the group of order 8 into itself, and in combination with the latter therefore generates one of the required groups of order 56 affecting 8 letters. As the latter can obviously be constructed in essentially only one way, it only remains to show that the substitutions of 9 letters can also be chosen in only one way, and that the result is actually a group.

For this purpose it is convenient to recur to the fact that the group of order 504 contains 36 subgroups of order 7, each of which is therefore transformed into itself by 14 substitutions. These form a non-cyclical group of this order which therefore contains 7 substitutions of order 2, which transform the corresponding substitution of order 7 into its 6th power. For the particular substitution

$$\sigma = (bcedghf),$$

one of the required substitutions of order 2 is

$$t = (cf)(eh)(dg)(ai),$$

and the combination of this with $u = (ac)(bd)(eg)(fh)$ gives

$$tu = \rho = (aichgbdef).$$

The last substitution combined with the group of order 56 above gives rise to a group the order of which is a multiple of $9.56 = 504$.

The group of order 56 being denoted by H, it can now be shown that for every α

$$\rho^\alpha H = H\rho^\beta.$$

The order of the group generated by ρ and H is then exactly 504. Writing

$$\begin{array}{lll} \tau_2 = (ab)(cd)(ef)(gh), & \tau_5 = (ae)(bf)(cg)(dh), & \tau_8 = (ah)(bg)(cf)(de), \\ \tau_3 = (ac)(bd)(eg)(fh), & \tau_6 = (af)(be)(ch)(dg), & \sigma = (bcedghf), \\ \tau_4 = (ad)(bc)(eh)(fg), & \tau_7 = (ag)(bh)(ce)(df), & \rho = (aichgbdef), \end{array}$$

and noticing that σ and any τ, as τ_3, suffice to generate H, we have only to show that for every α

$$\rho^\alpha \tau_3 = H\rho^\beta, \quad \rho^\alpha \sigma = H\rho^\gamma.$$

But we have at once

$$\tau_3^{-1}\rho\tau_3 = \rho^{-1}, \quad \rho^\alpha\tau_3 = \tau_3\rho^{-\alpha},$$

and a brief calculation furnishes

$$\begin{array}{ll} \rho\sigma = \sigma\tau_7\rho^6, & \rho^5\sigma = \tau_6\rho^3, \\ \rho^2\sigma = \sigma^6\rho^7, & \rho^6\sigma = \sigma^5\tau_6\rho^5, \\ \rho^3\sigma = \sigma^2\tau_4\rho^2, & \rho^7\sigma = \sigma^2\tau_3\rho^4, \\ \rho^4\sigma = \sigma^5\tau_3\rho, & \rho^8\sigma = \sigma^6\tau_4\rho^8. \end{array}$$

The existence of the group of order 504 having now been demonstrated, it is easy to show that this group is in reality simple. Those of its substitutions which affect less than 9 letters are already known. They are 216 substitutions of order 7 and 63 of order 2, the latter affecting 8 letters each. The remaining 224 substitutions affecting 9 letters must all be regular, and are therefore all of order 3 or order 9. They are all certainly contained in subgroups of order 9. But the number of these subgroups cannot exceed 28, and in this case they can furnish 224 substitutions only if they have no substitutions common to two of them. There are, therefore, in fact 28 subgroups of order 9. As one of these, $\{\rho\}$, is cyclical, they are all cyclical, and, as just shown, their substitutions are all different.

If, now, the group contained a self-conjugate subgroup, the latter must be transitive. Then its order must be a multiple of 9, and accordingly it would contain a subgroup of order 9 and therefore all the subgroups of order 9. Its order, being a divisor of 504, and being as great as 224, must therefore be 252. But the 27 remaining substitutions could not be taken either from those of order 2 or those of order 7; for if any one of either set occurs in a self-conjugate subgroup, all of the set must occur. There is, therefore, no self-conjugate subgroup. It appears then that

There is one, and only one, simple group of order 504. *This group can be expressed as a transitive substitution group of* 9 *letters, and in this form it can be generated by the substitutions*

$$(ab)(cd)(ef)(gh), \quad (bcedghf), \quad (aichgbdef).$$

It contains 28 *cyclical subgroups of order* 9, *having no operation except identity common to any two of them;* 36 *subgroups of order* 7*; and* 9 *subgroups of order* 8, *each consisting of* 7 *substitutions of order* 2 *and identity.*

IV.—*The Simple Group of Order* 660.

This group must contain exactly 12 subgroups of order 11. The isomorphic substitution group of 12 letters contains 12 subgroups of order 55 affecting 11 letters each. Each of these subgroups contains 11 conjugate subgroups of order 5 affecting 10 letters each. The total number of subgroups of order 5 in the entire group is therefore $\frac{12}{2} . 11 = 66$, and each of these is transformed into itself by a group of 10 substitutions.

These last groups cannot be cyclical, for there would be 66 of them, and they would furnish 66.4 new operations of order 10. There would then remain only 9 unidentified operations, and these could include at the most only 4 subgroups of order 3. Accordingly, each of the subgroups of order 10 contains 5 substitutions of order 2, which must here affect 12 letters each. These substitutions transform those of order 5 contained in the same group of order 10 into their 4th powers. If, now, the substitution of order 5 is, for example,

$$s = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10),$$

the corresponding substitutions of order 2, which must affect 12 letters and transform s into its 4th power, can only be

$$\begin{gathered}(1\ 6)\ (2\ 10)\ (3\ 9)\ (4\ 8)\ (5\ 7)\ (11\ 12),\\(1\ 7)\ (2\ 6)\ (3\ 10)\ (4\ 9)\ (5\ 8)\ (11\ 12),\\(1\ 8)\ (2\ 7)\ (3\ 6)\ (4\ 10)\ (5\ 9)\ (11\ 12),\\(1\ 9)\ (2\ 8)\ (3\ 7)\ (4\ 6)\ (5\ 10)\ (11\ 12),\\(1\ 10)\ (2\ 9)\ (3\ 8)\ (4\ 7)\ (5\ 6)\ (11\ 12).\end{gathered}$$

As soon, therefore, as a circular substitution of order 11 is assigned, one of order 5 is fixed and with it 5 of order 2. These are sufficient to generate the entire group, and there can therefore be only one type of a simple group of this order. That such a type exists is well known (cf. for example Klein, Elliptische Modulfunctionen, I, Chap. VIII).

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